FUZZY CONTINUOUS DYNAMICAL SYSTEM: A MULTIVARIATE OPTIMIZATION TECHNIQUE

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Abstract
This paper presents a multivariate optimization technique for the numerical simulation of continuous dynamical systems whose parameters, functional forms and/or initial conditions are modeled by fuzzy distributions. Fuzzy differential equation (FDE) is interpreted by using the strongly generalized differentiability concept and is shown that by this concept any FDE can be transformed to a system of ordinary differential equations (ODEs). By solving the associate ODEs one can find solutions for FDE. This approach has an inherited drawback of increasing uncertainty at each instance of time generally with nonlinear functional forms. Here we present a methodology to numerically simulate interval calculus and implements a new approach to the numerical integration of fuzzy dynamical systems, where the propagation of imprecision as a fuzzy distribution in the phase space is solved by a constrained multivariate optimization technique. Numerical simulations of some fuzzy dynamical systems (viz. Lotka Volterra model, Lorenz model) are also reported. Finally ecological degradation in wetlands of India is modeled by fuzzy initial value problem and some sustainable solution is proposed.
Keywords: Fuzzy continuous differential system, Numerical solution, Fuzzy initial value problem (FIVP), Multivariate optimization, Property of sufficiency of vertices (PSV), Ecological degradation model.

Fuzzy Dynamical Systems

Let us consider the FDE with initial value condition:

$$x'(t) = f(t,x)\ ,\ x(t_0) = x_0 \tag{1}$$

Where f: $[t_0,T]\times\mathbb{R}_F\rightarrow\mathbb{R}_F$ is a continuous fuzzy mapping and $x_0 \in\mathbb{R}_F$ and T is positive number or infinity with x_0 fuzzy initial condition defined on the n-dimensional domain Y. We interpret this notation as a fuzzy extension of an ordinary differential equation. We consider a fuzzy differential equation as a deterministic differential equation where some coefficients or initial condition are uncertain and represented in a possibilistic form: its solution is then the time evolution of a fuzzy region of uncertainty which Corresponds to the possibility distribution in the phase space.

Let us suppose α-cut of functions x(t), x_0, $f(t,x)$ are the following form:

$$[x(t)]^\alpha = \left[\underline{x_\alpha(t)}, \overline{x_\alpha(t)}\right]$$

$$[x_0]^\alpha = \left[\underline{x_0}, \overline{x_0}\right]$$

$$[f(t,x(t))]^\alpha = \left[\underline{f_\alpha}(t,\underline{x_\alpha},\overline{x_\alpha}), \overline{f_\alpha}(t,\underline{x_\alpha},\overline{x_\alpha})\right]$$

Then we have two following cases:

Case (I): If x(t) is 1-differentiable then solving FIVP (1) translates into the following algorithm:

Step (i) Solving the following system of ODEs:

$$\begin{aligned}\underline{x'_\alpha(t)} &= \underline{f_\alpha}(t,\underline{x_\alpha},\overline{x_\alpha}) = F(t,\underline{x},\overline{x}), \underline{x(t_0)} = \underline{x_0} \\ \overline{x'_\alpha(t)} &= \overline{f_\alpha}(t,\underline{x_\alpha},\overline{x_\alpha}) = G(t,\underline{x},\overline{x}), \overline{x(t_0)} = \overline{x_0}\end{aligned} \tag{2}$$

Step (ii) Ensure that the solution $[\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ and $[\underline{x'}_\alpha(t), \overline{x'}_\alpha(t)]$ are valid level sets.

Step (iii) By using the representation theorem again, we construct a 1-solution (t) such that $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$, for all $\alpha\in[0,1]$.

Case (II): If x(t) is 2-differentiable then solving FIVP (1) translates into the following algorithm:

Step (i) Solving the following system of ODEs:

$$\underline{x'_\alpha(t)} = \overline{f_\alpha}(t,\underline{x_\alpha},\overline{x_\alpha}) = G(t,\underline{x},\overline{x}), \underline{x(t_0)} = \underline{x_0} \tag{3}$$

$$\overline{x'_\alpha(t)} = \underline{f_\alpha}(t, \underline{x_\alpha}, \overline{x_\alpha}) = F(t, \underline{x}, \overline{x}), \overline{x(t_0)} = \overline{x_0}$$

Step (ii) Ensure that the solution $[\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ and $[\underline{x'}_\alpha(t), \overline{x'}_\alpha(t)]$ are valid level sets.
Step (iii) By using the representation theorem again, we construct a 1-solution $x(t)$ such that $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$, for all $\alpha \in [0,1]$.

Problem to Solve Numerically

To solve numerically the system (2) and (3) it might seem intuitive to apply the interval mathematics (Moore, 1966) directly to the numerical algorithms for deterministic systems. However,(Bonarini, Bontempi, [25]) showed that a straightforward use of interval mathematics for the resolution of fuzzy (or interval) differential equation can produce incorrect results. This is due to the fact that the fuzzy (and interval) formalism is unable to represent the interaction that the differential equation establishes between variables: as a consequence, spurious values are introduced into the solution and the evolution of the system may reach region where no numerical solution exists. Figure 1 illustrates the problem. The figure plots the evolution of the initial condition (i.e. the rectangle ABCD) in a oscillatory 2nd order system: if we represent the uncertain state of the dynamical system at time t' in the interval formalism, we introduce in the solution spurious points (black regions) which are not the evolution of points belonging to ABCD.

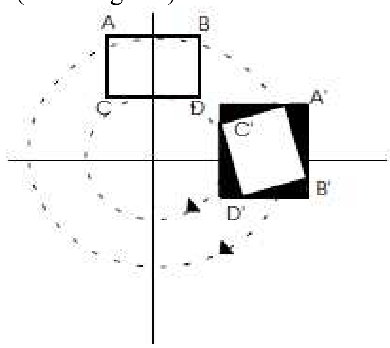

Fig. 1 Introduction of spurious trajectories in the non interacting simulation of an oscillating system

This approach is instead the following: the initial condition of the interval differential equation defines a hyper cube. We call this n-cube the *region of uncertainty* of the system at time t=0. To solve the IDE means to compute the evolution in time of the region of uncertainty.
Bonarini and Bontempi proved that under general conditions of continuity and differentiability it is sufficient to compute the evolution of the external surface of the uncertainty region (Bonarini, Bontempi, [25]). This leads to the conclusion that it is sufficient to calculate the trajectories of the points belonging to the external surface of the region of uncertainty to know the evolution in time of the region itself. However, the quantity of trajectories to be computed is still infinite, also if it is of a lower order.

Numerical Solution Algorithm using Property of Sufficiency of Vertices

The problem remains how to sample, in the most convenient way, the external surface of the region of uncertainty during its time evolution. Let us remind that the performance of a multivariable optimization algorithm may be greatly increased by providing to it the partial derivatives (gradient) of the function value respect to the arguments: in our case we should provide the derivatives at time t* of x_i respect to $\mathbf{x_0} \in \mathbf{x_{0\alpha}}$. To obtain these derivatives we use the connection matrix (Moore, 1966), as follows. We denote by $\mathbf{x}(t, \mathbf{x_0})$ the n-dimensional, real vector solution to the numeric initial value problem. We define the connection matrix for the solution $\mathbf{x}(t, \mathbf{x_0})$ as the matrix $C(t, \mathbf{x_0})$ with elements

$$C_{ij} = \frac{\partial x_i(t,x)}{\partial x_j} \Big|_{x = \boldsymbol{x_0}} \qquad (5)$$

We denote by $J(t,\mathbf{x_0})$ the Jacobian matrix of the vector function evaluated at $\mathbf{x}(t,\mathbf{x_0})$.
The matrix $J(t, \mathbf{x_0})$ has elements

$$J_{ij}(t, \mathbf{x_0}) = \frac{\partial f_i(t,x)}{\partial x_j} \Big|_{x = x(t, \boldsymbol{x_0})} \qquad (6)$$

Moore (1966) demonstrated that

$$\frac{\partial C(t,\mathbf{x_0})}{\partial t} = J(t, \mathbf{x_0}) \, .C(t, \mathbf{x_0}) \qquad (7)$$

with $C(0, \mathbf{x_0}) = I$ where I is the identity matrix.

Now in the next time state the evolution of the surface points of region of uncertainty will generate new region of uncertainty. So property of sufficiency of vertices is used to identify surface points and reduce the order of the sample space. Let us consider a generic parametric line L in the n-D space whose equations are

$$x_i = x_i(v) \quad v \in [v_0, v_1],\ i = 1,2,\ldots\ldots,n.$$

If $x_i(v)$, i = 1,2,….,n are monotonous functions of v, it is sufficient to know the coordinates ($x_1(v_0)$, $x_2(v_0)$, ….,$x_n(v_0)$) and ($x_1(v_1)$, $x_2(v_1)$, ….,$x_n(v_1)$) of the extremes P_0 and P_1 of L to compute $x_i(v)$ i = 1, 2, ….,n. We say that L satisfies the *property of sufficiency of the vertices (p.s.v) if it is* sufficient to know the values of L at the extremes P0 and P1 to compute the intervals. The following necessary condition for the p.s.v. holds: for any pair of points $p_{0k} \in L_{v0}$ and $p_{1j} \in L_{v1}$ the total derivatives with respect to v have the same sign, that is: $\frac{\partial x_i}{\partial v}(p_{1i}).\frac{\partial x_i}{\partial v}(p_{0k}) > 0; \forall i = 1,2,\ldots\ldots,n$

Thus if $\frac{\partial x_i}{\partial v}(p_{1i}).\frac{\partial x_i}{\partial v}(p_{0k}) < 0$ we could conclude that there exist a point of extremum in $[p_{0k}, p_{1i}]$.

If $\frac{\partial x_i}{\partial v}(p_{1i}).\frac{\partial x_i}{\partial v}(p_{0k}) = 0$ one of these points is an extremum.

If $\frac{\partial x_i}{\partial v}(p_{1i}).\frac{\partial x_i}{\partial v}(p_{0k}) > 0$ we should proceed the iteration through the line v.

The elements of the matrix give the sensitivity of the solution components $x_i(t, \mathbf{x_0})$ with respect to small changes of the initial values $(\mathbf{x_0})_i$. By coupling the IDE system with the set of n^2 equations coming from the connection matrix differential system we obtain an augmented differential system which gives at time t* the values of the variables x_i and the values of the derivatives of the variables x_i respect to the initial condition $\mathbf{x_0}$. We may then use these combined values as the value and the gradient of the function to be maximized (or minimized).

Let $\Omega \in \mathbb{R}^n$ be the region of uncertainty in the phase space at a time t^* and $\overrightarrow{x_0} \in \Omega$ be the initial value vector. Then to get the region of uncertainty at any time instance t we need to solve basically the following multivariate optimization problem

For each state variable x_i :

Maximize (minimize) $x_i(t, \overrightarrow{x_0})$

Subject to

$\overrightarrow{x_0} \in \Omega$.

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REFERENCES

1. S. Abbasbandy, T. Allahvinloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computational Methods in applied Mathematics 2, 113-124, 2002.
2. S. Abbasbandy, T. Allahvinloo, O. Lopez-Pouso, J.J Nieto, Numerical methods for fuzzy differential inclusions, Journal of Computer and Mathematics with Applications 48, 1633-1641, 2004.
3. T. Allahvinloo, N. Ahmadi, E. Ahmadi, Numerical solution of fuzzy differential equations by predictor-corrector method, Information Sciences 177, 1633-1647, 2007.
4. B. Bede, Note on "Numerical solutions of fuzzy differential equations by predictor-corrector method", Information Sciences, 178, 1917-1922, 2008.
5. B. Bede., S.G Gal, Almost periodic fuzzy-number-valued functions, Fuzzy Sets and Systems 147, 385-403, 2004.
6. B. Bede., S.G Gal S.G., Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151, 581-599, 2005.
7. J.C. Butcher, Numerical methods for ordinary differential equations, John Wiley & Sons, Great Britain, 2003.
8. Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations,Chaos, Solitons and Fractals 38, 112-119, 2008.
9. S.L Chang, L.A. Zadeh, On fuzzy mapping and control, IEEE Trans, Systems Man Cybernet. 2, 30-34, 1972.
10. P. Diamond, Stability and periodicity in fuzzy differential equations, IEEE Trans. Fuzzy Systems 8, 583-590, 2000.

11. D. Dubois, H. Prade, Towards fuzzy differential calculus: part 3, differentiation, Fuzzy Sets and Systems 8, 225-233, 1982.
12. R. Goetshel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18, 31-43, 1986.
13. E. Hullermeier, An approach to modeling and simulation of uncertain dynamical systems, Internat. J. Uncertainty Fuzzyness Knowledge-Based Systems 5, 117-137, 1997.
14. O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24, 301-317, 1987.
15. O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems 35, 389-396, 1990.
16. A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations by Nystrom method, Chaos, Solitons& Fractals 41, 859-868, 2009.
17. M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems, 105, 133-138, 1999.
18. J.J. Nieto, A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis:Hybrid System 3, 700-707, 2009.
19. M.L. Puri, D. Ralescu, Differentials of fuzzy functions, J. Math.Anal.Appl, 91, 321-325, 1983.
20. S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24, 319-330, 1987.
21. C. Song, S. Wu, Existence and uniqueness of solutions to the Cauchy problem of fuzzy differential equations, Fuzzy Sets and Systems 110, 55-57, 2000.
22. Z. Akbarzadeh Ghanaie1, M. MohseniMoghadam Solving fuzzy differential equations by Runge-Kutta method The Journal of Mathematics and Computer Science Vol .2 No.2 (2011) 208-221
23. D. Berleant, B. Kuipers Qualitative-Numeric Simulation with Q3 in *RecentAdvances in Qualitative Physics*, BoiFaltings and Peter Struss editors, The MITPress, Cambridge, (1992)
24. A. Bonarini and G. Bontempi. Qua.Si.: A qualitative simulation approach for fuzzy models, in A.Guasch and R.M. Huber (Eds.) *Modeling and Simulation ESM 94 (Proceedings European Simulation Multiconference 1994*), SCSInternational, Gent, Belgium, 420—424 (1994a)
25. A. Bonarini and G. Bontempi. A qualitative simulation approach for fuzzy dynamical models.*ACM Transactions on Modeling and Computer Simulation*(TOMACS), 4, 4, (1994b).
26. E.N. Lorenz Deterministic non periodic flow. *J. Atmos. Sci*., vol. 20, pp. 130-141 (1983).
27. R.E. Moore. *Interval Analysis*, Prenctice-Hall, Englewood Cliffs, NJ, (1966).
28. H.T. Nguyen. A note on the Extension Principle for Fuzzy Sets.*Journal ofMathematical Analysis Applications*, 64, 2, (1978), 369-380.
29. K. Chopra ,S.K. Adhikari, Modeling a wetland system for ecological & economical value. Environment and development Economics, vol9, Cambridge university press, UK, 19-45.
30. V. Rai, Modeling a wetland system: Case Study Keoladeo National Park,(KNP) India. Ecological modeling 210 (2008), 247-252
31. M. Bandyopadhyay and C.G. Chakraborti, Deterministic and stochastic analysis of non-linear prey-predator systems.J.Biol. Systems II (2003), n0 2, 161-172.
32. G. Shafer. *A mathematical theory of evidence*, Princeton University Press, Princeton, NJ, (1976)
33. D. Bobrow. ed. *Qualitative Reasoning about Physical systems*, MIT Press, Cambridge, Massachusetts, (1984)
34. L.A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1, 1, (1978), 3-28.
35. G.J. Klir and T.A. Folger. *Fuzzy sets, uncertainty, and information*, Prentice- Hall, New York (1988).
36. N. Bouleau, D. Lepingle. *Numerical methods for stochastic processes*, John Wiley, New York, (1994)
37. W.H. Press, B. Flannery, S.A. Teukolsky. and W.T. Vetterling. *Numerical Recipes: the art of scientific computing*, CUP, Cambridge, (1986).