Multiple solutions for quasilinear equations depending on a parameter

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Abstract

The purpose of this talk is to use a very recent three critical points theorem due to Bonanno and Marano [1] to establish the existence of at least three solutions for quasilinear second order differential equations on a compact interval $[a, b] \subset R$ under appropriate hypotheses. We exhibit the existence of at least three (weak) solutions and, and the results are illustrated by examples.

Keywords- Dirichlet problem; Critical point; Three solutions; Multiplicity results.
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1 Main results

Consider the following quasilinear two-point boundary value problem

$$\begin{cases} -u'' = (\lambda f(x, u) + g(u))h(u') & \text{in } (a, b), \\ u(a) = u(b) = 0 \end{cases} \tag{1}$$

where $[a, b] \subset R$ is a compact interval, $f : [a, b] \times R \to R$ is an L^1-Carat$\acute{e}$odory function, $g : R \to R$ is a Lipschitz continuous function with $g(0) = 0$, i.e., there exists a constant $L \geq 0$ provided

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for all $t_1, t_2 \in R$, $h : R \to]0, +\infty[$ is a bounded and continuous function with $m := \inf h > 0$ and λ is a positive parameter.

Employing Theorem 3.6 of [1], we establish the existence of at least three distinct (weak) solutions in $W_0^{1,2}([a, b])$ to the problem (1) for any fixed positive parameter λ belonging to an exact interval which will be observed in the main results.

We mean by a (weak) solution of problem (1), any $u \in W_0^{1,2}([a,b])$ such that

$$\int_a^b u'(x)v'(x)dx - \int_a^b [\lambda f(x,u(x)) + g(u(x))]h(u'(x))v(x)dx = 0$$

for every $v \in W_0^{1,2}([a,b])$. Denote $M := \sup h$ and suppose that the constant $L \geq 0$ satisfies $LM(b-a)^2 < 4$.

We introduce the functions $F : [a,b] \times R \to R$, $H : R \to R$ and $G : R \to R$ respectively, as follows

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [a,b] \times R,$$

$$H(t) = \int_0^t \int_0^\tau \frac{1}{h(\delta)} d\delta d\tau \quad \text{for all } t \in R$$

and

$$G(t) = -\int_0^t g(\xi)d\xi \quad \text{for all } t \in R.$$

We now formulate our main result.

Theorem 1. Assume that there exist a positive constant r and a function $w \in W_0^{1,2}([a,b])$ such that

(α_1) $\int_a^b [G(w(x)) + H(w'(x))]dx > r$,

(α_2) $\dfrac{\int_a^b \sup_{t \in [-\sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}, \sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}]} F(x,t)dx}{r} < \dfrac{\int_a^b F(x,w(x))dx}{\int_a^b [G(w(x))+H(w'(x))]dx}$,

(α_3) $\limsup_{|t| \to +\infty} \frac{F(x,t)}{t^2} < \frac{4-LM(b-a)^2}{2M(b-a)^2 r} \int_a^b \sup_{t \in [-\sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}, \sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}]} F(x,t)dx$ uniformly respect to $x \in [a,b]$.

Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{\int_a^b [G(w(x)) + H(w'(x))]dx}{\int_a^b F(x,w(x))dx}, \frac{r}{\int_a^b \sup_{t \in [-\sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}, \sqrt{\frac{2M(b-a)r}{4-LM(b-a)^2}}]} F(x,t)dx} \right[$$

the problem (1) admits at least three distinct weak solutions in $W_0^{1,2}([a,b])$.

References

[1]G. Bonanno, S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition,* Appl. Anal. 89 (2010) 1-10.