Some Topological and Geometric Properties of the Domain of the Double Sequential Band Matrix $B(\widetilde{r},\widetilde{s})$ in the Sequence Space $\ell(p)$

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Abstract. The sequence space $\ell(p)$ was introduced by Maddox [Spaces of strongly summable sequences, Quart. J. Math. Oxford (2)**18**(1967), 345–355]. In the present paper, the sequence space $\ell(\widetilde{B},p)$ of non-absolute type, the domain of the double sequential band matrix $B(\widetilde{r},\widetilde{s})$ in the sequence space $\ell(p)$, is introduced. Furthermore, the α-, β- and γ-duals of the space $\ell(\widetilde{B},p)$ are determined, and the Schauder basis is given. The classes of matrix transformations from the space $\ell(\widetilde{B},p)$ to the spaces ℓ_∞, f and c are characterized. Additionally, the characterizations of some other matrix transformations from the space $\ell(\widetilde{B},p)$ to the Euler, Riesz, difference, etc., sequence spaces are obtained by means of a given lemma. Finally, some properties of the space $\ell(\widetilde{B},p)$ are examined.

Keywords: Paranormed sequence space, double sequential band matrix, α-, β- and γ-duals, matrix transformations and rotundity of a sequence space.
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PRELIMINARIES, BACKGROUND AND NOTATION

By w, we denote the space of all real valued sequences. Any vector subspace of w is called a *sequence space*. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively; where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1,H\}$. Then, the linear spaces $\ell(p)$ was defined by Maddox [21] (see also Simons [25] and Nakano [24]) as $\ell(p) = \{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\}$, $(0 < p_k \le H < \infty)$ which is the complete space paranormed by $g(x) = (\sum_k |x_k|^{p_k})^{1/M}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout that $p_k^{-1} + (p_k')^{-1} = 1$ provided $\inf p_k \le H < \infty$ and denote the collection of all finite subsets of $\mathbb{N} = \{0,1,2,\dots\}$ by $\mathscr{F}$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n,k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

The main purpose of this paper, which is a continuation of Kirişçi and Başar [18], is to introduce the sequence space $\ell(\widetilde{B},p)$ of non-absolute type consisting of all sequences whose $B(\widetilde{r},\widetilde{s})$-transforms are in the space $\ell(p)$; where the double sequential band matrix $B(\widetilde{r},\widetilde{s}) = \{b_{nk}(r_k,s_k)\}$ is defined by $b_{nk}(r_k,s_k) = \begin{cases} r_k &, \quad k = n, \\ s_k &, \quad k = n-1, \\ 0 &, \quad \text{otherwise} \end{cases}$ for all

$k,n \in \mathbb{N}$; where $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ are the convergent sequences whose entries either constants or distinct real numbers. Furthermore, the basis is constructed and the α-, β- and γ-duals are computed for the space $\ell(\widetilde{B},p)$. Besides this, the matrix transformations from the space $\ell(\widetilde{B},p)$ to some sequence spaces are characterized. Finally, some results related to the rotundity of the space $\ell(\widetilde{B},p)$ are derived.

It is clear that $\Delta^{(1)}$ can be obtained as a special case of $B(\widetilde{r},\widetilde{s})$ for $\widetilde{r} = e$ and $\widetilde{s} = -e$ and it is also trivial that $B(\widetilde{r},\widetilde{s})$ is reduced in the special case $\widetilde{r} = re$ and $\widetilde{s} = -se$ to the generalized difference matrix $B(r,s)$. So, the results related to the matrix domain of the matrix $B(\widetilde{r},\widetilde{s})$ are more general and more comprehensive than the corresponding consequences of the matrix domains of $\Delta^{(1)}$ and $B(r,s)$.

THE SEQUENCE SPACE $\ell(\widetilde{B},p)$ OF NON-ABSOLUTE TYPE

In this section, we introduce the complete paranormed linear space $\ell(\widetilde{B},p)$.

The *matrix domain* λ_A of an infinite matrix A in a sequence space λ is defined by $\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}$. Choudhary and Mishra [13] defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that S-transforms of them are in the space $\ell(p)$, where $S = (s_{nk})$ is defined by $s_{nk} = 1$, if $0 \le k \le n$ and $s_{nk} = 0$, otherwise. Başar and Altay [10] have recently examined the space $bs(p)$ which is formerly defined by Başar in [9] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Aydın and Başar [8] have studied the space $a^r(u,p)$ which is the domain of the matrix A^r in the sequence space $\ell(p)$, where the matrix $A^r = \{a_{nk}(r)\}$ is defined by $a_{nk}(r) = (1+r^k)u_k/(n+1)$, if $0 \le k \le n$ and $a_{nk}(r) = 0$, otherwise; where $(u_k) \in w$ with $u_k \neq 0$ for all $k \in \mathbb{N}$ and $0 < r < 1$. Altay and Başar [2] have studied the sequence space $r^t(p)$ which is derived from the sequence space $\ell(p)$ of Maddox by the Riesz means R^t. Following Choudhary and Mishra [13], Başar and Altay [10], Altay and Başar [2, 4, 5, 6], Aydın and Başar [7, 8], we introduce the sequence space $\ell(\widetilde{B},p)$ as the set of all sequences whose $B(\widetilde{r},\widetilde{s})$-transforms are in the space $\ell(p)$, that is

$$\ell(\widetilde{B},p) := \left\{ (x_k) \in w : \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k} < \infty \right\}, \quad (0 < p_k \le H < \infty).$$

It is trivial that in the case $p_k = p$ for all $k \in \mathbb{N}$, the sequence space $\ell(\widetilde{B},p)$ is reduced to the sequence space $\widetilde{\ell}_p$ which is introduced by Kirişçi and Başar [18]. Define the sequence $y = (y_k)$ as the $B(\widetilde{r},\widetilde{s})$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = (\widetilde{B}x)_k = r_k x_k + s_{k-1}x_{k-1} \quad \text{for all } k \in \mathbb{N}. \tag{2}$$

Since the spaces $\ell(p)$ and $\ell(\widetilde{B},p)$ are linearly isomorphic one can easily observe that $x = (x_k) \in \ell(\widetilde{B},p)$ if and only if $y = (y_k) \in \ell(p)$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (2).

Now, we may begin with the following theorem without proof which is essential in the text:

Theorem 1 *$\ell(\widetilde{B},p)$ is a complete linear metric space paranormed by the paranorm $h(x) = (\sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k})^{1/M}$, where $M = \max\{1, \sup p_k\}$ and $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.*

Therefore, one can easily check that the absolute property does not hold on the space $\ell(\widetilde{B},p)$, that is $h(x) \neq h(|x|)$; where $|x| = (|x_k|)$. This says that $\ell(\widetilde{B},p)$ is the sequence space of non-absolute type.

Theorem 2 *Convergence in $\ell(\widetilde{B},p)$ is strictly stronger than coordinatewise convergence, but the converse is not true.*

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field. A K-space λ is called an *FK-space* provided λ is complete linear metric space. An FK-space whose topology is normable is called a *BK-space*. Now, we may give the following:

Theorem 3 *$\widetilde{\ell}_p$ is the linear space under the coordinatewise addition and scalar multiplication which is the BK-space with the norm $\|x\| := (\sum_k |s_{k-1}x_{k-1} + r_k x_k|^p)^{1/p}$, where $1 \le p < \infty$.*

With the notation of (2), define the transformation T from $\ell(\widetilde{B},p)$ to $\ell(p)$ by $x \mapsto y = Tx$. Since T is a linear bijection, we have

Corollary 4 *The sequence space $\ell(\widetilde{B},p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

Theorem 5 *The space $\ell(\widetilde{B},p)$ has AK.*

Since, it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [16, Remark 2.4]), we have:

Corollary 6 *Let $0 < p_k \leq H < \infty$ and $\alpha_k = \{B(\widetilde{r},\widetilde{s})x\}_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(r,s) = \left\{b_n^{(k)}(r,s)\right\}_{n\in\mathbb{N}}$ of the elements of the space $\ell(\widetilde{B},p)$ by $b_n^{(k)} := \begin{cases} \frac{(-1)^{n-k}}{r_n}\prod_{j=k}^{n-1}\frac{s_j}{r_j} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases}$ for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}(r,s)\}_{k\in\mathbb{N}}$ is a basis for the space $\ell(\widetilde{B},p)$ and any $x \in \ell(\widetilde{B},p)$ has a unique representation of the form $x := \sum_k \alpha_k b^{(k)}$.*

THE α-, β- AND γ-DUALS OF THE SPACE $\ell(\widetilde{B},p)$

In this section, we give the theorems determining the α-, β- and γ-duals of the sequence space $\ell(\widetilde{B},p)$ of non-absolute type.

The set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) := \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\right\} \tag{3}$$

is called the *multiplier space* of the sequence spaces λ and μ. With the notation of (3), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha := S(\lambda,\ell_1), \quad \lambda^\beta := S(\lambda,cs) \quad \text{and} \quad \lambda^\gamma := S(\lambda,bs).$$

Theorem 7 *Define the sets $S_1(p)$ and $S_2(p)$ by*

$$S_1(p) = \bigcup_{B>1}\left\{a = (a_k) \in w : \sup_{N\in F}\sum_k\left|\sum_{n\in N}\frac{(-1)^{n-k}}{r_n}\prod_{j=k}^{n-1}\frac{s_j}{r_j}a_n B^{-1}\right|^{p_k'} < \infty\right\},$$

$$S_2(p) = \left\{a = (a_k) \in w : \sup_{N\in F}\sup_{k\in\mathbb{N}}\left|\sum_{n\in N}\frac{(-1)^{n-k}}{r_n}\prod_{j=k}^{n-1}\frac{s_j}{r_j}a_n\right|^{p_k} < \infty\right\}.$$

Then, $\{\ell(\widetilde{B},p)\}^\alpha = \begin{cases} S_1(p) & , \quad 1 < p_k \leq H < \infty \text{ for all } k \in \mathbb{N}, \\ S_2(p) & , \quad 0 < p_k \leq 1 \text{ for all } k \in \mathbb{N}. \end{cases}$

Theorem 8 *Define the sets $S_3(p)$, $S_4(p)$ and $S_5(p)$ by*

$$S_3(p) = \bigcup_{B>1}\left\{a = (a_k) \in w : \sup_n\sum_k\left|\sum_{i=0}^{n-k}\frac{(-1)^i}{r_{i+k}}\prod_{j=k}^{i+k-1}\frac{s_j}{r_j}a_{i+k}B^{-1}\right|^{p_k'} < \infty\right\},$$

$$S_4(p) = \left\{a = (a_k) \in w : \sum_{i=0}^{\infty}\frac{(-1)^i}{r_{i+k}}\prod_{j=k}^{i+k-1}\frac{s_j}{r_j}a_{i+k} < \infty\right\},$$

$$S_5(p) = \left\{a = (a_k) \in w : \sup_{n,k\in\mathbb{N}}\left|\sum_{i=0}^{n-k}\frac{(-1)^i}{r_{i+k}}\prod_{j=k}^{i+k-1}\frac{s_j}{r_j}a_{i+k}\right|^{p_k} < \infty\right\}.$$

Then, $\{\ell(\widetilde{B},p)\}^\beta = \begin{cases} S_3(p) \cap S_4(p) & , \quad 1 < p_k \leq H < \infty \text{ for all } k \in \mathbb{N}, \\ S_4(p) \cap S_5(p) & , \quad 0 < p_k \leq 1 \text{ for all } k \in \mathbb{N}. \end{cases}$

Theorem 9 $\{\ell(\widetilde{B},p)\}^{\gamma} = \begin{cases} S_3(p) & , \quad 1 < p_k \leq H < \infty \text{ for all } k \in \mathbb{N}, \\ S_5(p) & , \quad 0 < p_k \leq 1 \text{ for all } k \in \mathbb{N}. \end{cases}$

MATRIX TRANSFORMATIONS ON THE SEQUENCE SPACE $\ell(\widetilde{B},p)$

In this section, we characterize the classes $(\ell(\widetilde{B},p):\ell_\infty)$, $(\ell(\widetilde{B},p):f)$ and $(\ell(\widetilde{B},p):c)$ of matrix transformations. We consider only the case $1 < p_k \leq H < \infty$ and leave the case $0 < p_k \leq 1$ to the reader because of it can be proved in the similar way.

We write for brevity that $\widetilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} a_{nj}$ for all $k,n \in \mathbb{N}$.

Theorem 10 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B},p):\ell_\infty)$ if and only if and there exists an integer $M > 1$ such that

$$\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{i=0}^{n-k} \frac{(-1)^i}{r_{i+k}} \prod_{j=k}^{i+k-1} \frac{s_j}{r_j} a_{n,i+k} B^{-1} \right|^{p_k'} < \infty, \tag{4}$$

$$\sum_i \frac{(-1)^i}{r_{i+k}} \prod_{j=k}^{i+k-1} \frac{s_j}{r_j} a_{n,i+k} < \infty. \tag{5}$$

(ii) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B},p):\ell_\infty)$ if and only if the condition (5) holds

$$\sup_{n,k\in\mathbb{N}} \left| \sum_{i=0}^{n-k} \frac{(-1)^i}{r_{i+k}} \prod_{j=k}^{i+k-1} \frac{s_j}{r_j} a_{n,i+k} \right|^{p_k} < \infty. \tag{6}$$

Theorem 11 *Let the entries of the matrices $E = (e_{nk})$ and $F = (f_{nk})$ are connected with the relation*

$$e_{nk} := s_{k-1} f_{n,k-1} + r_k f_{nk} \quad or \quad f_{nk} := \sum_{i=k}^{\infty} \frac{(-1)^i}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} e_{ni} \tag{7}$$

for all $k,n \in \mathbb{N}$. Then, $E \in (\ell(\widetilde{B},p):f)$ if and only if $F \in (\ell(p):f)$ and $F^n \in (\ell(p):c)$ for every fixed $n \in \mathbb{N}$, where $F^n = \left(f_{mk}^{(n)}\right)$ *with* $f_{mk}^{(n)} := \begin{cases} \sum_{i=k}^{m} \frac{(-1)^i}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} e_{ni} & , \quad 0 \leq k \leq m, \\ 0 & , \quad k > m, \end{cases}$ *for all $m,k \in \mathbb{N}$.*

Theorem 12 *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B},p):c)$ if and only if (4)-(6) hold and $\lim_{n\to\infty} \widetilde{a}_{nk} = \alpha_k$ for every fixed $k \in \mathbb{N}$.*

SOME PROPERTIES OF THE SPACE $\ell(\widetilde{B},p)$

Among many geometric properties, the rotundity of Banach spaces is one of the most important topics in functional analysis. For details, the reader may refer to [12], [14] and [23]. In this section, we characterize the rotundity of the space $\ell(\widetilde{B},p)$ and emphasize some results related to this concept.

By $S(X)$ and $B(X)$, we denote the unit sphere and unit ball of a Banach space X, respectively. A point $x \in S(X)$ is called an *extreme point* if $2x = y + z$ implies $y = z$ for all $y,z \in S(X)$.

A Banach space X is said to be *rotund* (*strictly convex*) if every point of $S(X)$ is an extreme point.

Theorem 13 *The modular σ_p on $\ell(\widetilde{B},p)$ satisfies the following properties with $p_k \geq 1$ for all $k \in \mathbb{N}$*

(i) If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(\frac{x}{\alpha}) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha \sigma_p(x)$.

(ii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(\frac{x}{\alpha})$.

(iii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha \sigma_p(\frac{x}{\alpha})$.

Theorem 14 *For any $x \in \ell(\widetilde{B}, p)$, we have*

(i) If $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$,
(ii) If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$,
(iii) $\|x\| = 1$ if and only if $\sigma_p(x) = 1$,
(iv) $\|x\| < 1$ if and only if $\sigma_p(x) < 1$,
(v) $\|x\| > 1$ if and only if $\sigma_p(x) > 1$.

Theorem 15 *$\ell(\widetilde{B}, p)$ is a Banach space with Luxemburg norm.*

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