A Characterization of Compactness in Banach Spaces with Continuous Linear Representations of the Rotation Group of a Circle.

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Abstract

Let $\mathbb{H}$ be a complex Banach space, $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, $SO(2)$ be the group of all rotations of $\mathbb{T}$, $GL(\mathbb{H})$ be group of all invertible bounded linear operators on $\mathbb{H}$, $\alpha : SO(2) \to GL(\mathbb{H})$ be a continuous linear representation, $x \in \mathbb{H}$. For all $n \in \mathbb{Z}$, n-th Fourier coefficient of x with respect to the α is defined by

$$P_n(x) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-int} \alpha(t)(x) dt$$

and the Fourier series of x with respect to the α is defined by

$$\sum_{n=-\infty}^{+\infty} P_n(x). \tag{1}$$

The convergence of this series and some properties of $P_n(x)$ are investigated in [5]. In this work, a characterization of compactness in Banach space $\mathbb{H}$ is given by means of Fourier coefficients $P_n(x)$. One of the main results is as follows:

Theorem : Suppose that $dimH_n < +\infty$ for all $n \in \mathbb{Z}$. Then a closed subset $A \subset \mathbb{H}$ is compact if and only if for any $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that$\|\frac{n}{n+1}\sigma_n(x) - x\| < \varepsilon$ for all $x \in A$ and $n \geq N(\varepsilon)$.

Where, for all $n \in \mathbb{N} \cup \{0\}$, $\sigma_n(.) : \mathbb{H} \to \mathbb{H}$ is a linear bounded operator which is defined by

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x)$$

for all $x \in \mathbb{H}$, $S_k(x)$ is the k-th partial sum of (1) for all $k \in \mathbb{N} \cup \{0\}$ and

$$H_n := \{x \in \mathbb{H} : \alpha(t)(x) = e^{int}x, \forall t \in \mathbb{T}\}$$

for all $n \in \mathbb{Z}$.

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